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Two-dimensional $\mathcal{N} = (2,2)$ super Yang-Mills theory on computer

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Abstract: We carry out preliminary numerical study of Sugino's lattice formulation [1, 2] of the two-dimensional $\mathcal{N} = (2,2)$ super Yang-Mills theory (2d $\mathcal{N} = (2,2)$ SYM) with the gauge group SU(2). The effect of dynamical fermions is included by re-weighting a quenched ensemble by the pfaffian factor. It appears that the complex phase of the pfaffian due to lattice artifacts and flat directions of the classical potential are not problematic in Monte Carlo simulation. Various one-point supersymmetric Ward-Takahashi (WT) identities are examined for lattice spacings up to $a = 0.5/g$ with the fixed physical lattice size $L = 4.0/g$, where g denotes the gauge coupling constant in two dimensions. WT identities implied by an exact fermionic symmetry of the formulation are confirmed in fair accuracy and, for most of these identities, the quantum effect of dynamical fermions is clearly observed. For WT identities expected only in the continuum limit, the results seem to be consistent with the behavior expected from supersymmetry, although we do not see clear distintion from the quenched simulation. We measure also the expectation values of renormalized gauge-invariant bi-linear operators of scalar fields.

Keywords: Extended Supersymmetry, Lattice Gauge Field Theories, Field Theories in Lower Dimensions, Renormalization Regularization and Renormalons.

Contents

1. Introduction

It will be very exciting if non-perturbative question in supersymmetric gauge theories (such as possibility of spontaneous breaking of supersymmetry) can be studied numerically at one's will. Despite the great efforts being made towards numerical study of the four-dimensional $\mathcal{N} = 1$ super Yang-Mills theory (4d $\mathcal{N} = 1$ SYM) [3]–[9], so far no conclusive evidence of a restoration of supersymmetry in the continuum limit has been observed. For recent reviews on lattice formulation of supersymmetric theories, see refs. [10 – 12]. Under this situation, to test various ideas, it seems useful to examine lower dimensional supersymmetric gauge theories in great detail, which have much simpler ultraviolet (UV) structure and for which it is relatively easy to accumulate high statistics in Monte Carlo simulation.

In this paper, we report the results of our small-scale Monte Carlo study of lattice formulation of the 2d $\mathcal{N} = (2, 2)$ SYM, proposed by Sugino [1, 2]. For this and similar lower-dimensional supersymmetric gauge theories, many other proposals and studies on possible lattice formulation exist [13]–[25]. (See also ref. [26] for studies based on the supersymmetric discrete light-cone quantization.) The advantage of the formulation of refs. [1, 2] is that a fermionic symmetry, associated with one of four supercharges of the target theory, is manifestly preserved even with finite lattice spacings and finite volume. Full supersymmetry is expected to be restored in the continuum limit. Possible disadvantage of the formulation, on the other hand, is that the pfaffian resulting from the integration over fermionic fields is generally complex,[1] although the complex phase is expected to be

[1] To avoid this point is one of motivations of the proposal of ref. [19].

irrelevant in the continuum limit, as the corresponding pfaffian in the target theory is real and positive semi-definite.

In our simulation, we include the effect of dynamical fermions by re-weighting. That is, in taking a statistical average, a quenched ensemble is re-weighted by the factor of pfaffian. With parameters and statistics of our Monte Carlo simulation, it appears that the complex phase of the pfaffian and flat directions of the classical potential (which might imply subtlety in the integration over scalar fields) are not problematic. The parameters of our simulation correspond to lattice spacings up to $a = 0.5/g$ with the fixed physical lattice size $L = 4.0/g$, where g denotes the gauge coupling constant in two dimensions.

In this paper, we mainly study one-point supersymmetric bare WT identities. These are precisely WT identities numerically analysed by Catterall [22] on the basis of his lattice formulation of the 2d $\mathcal{N} = (2,2)$ SYM [16]. In our numerical simulation, WT identities implied by the exact fermionic symmetry of the formulation are reproduced in fair accuracy and, for most of these identities, we clearly observe the quantum effect of dynamical fermions. For WT identities expected only in the continuum limit, the results seem to be consistent with the behavior expected by supersymmetry, although we do not see clear distinction from the quenched (i.e., non supersymmetric) simulation. We measure also the expectation values of renormalized gauge-invariant bi-linear operators of scalar fields to illustrate how this kind of numerical study would be useful.

In section 2, we briefly review Sugino's formulation of the 2d $\mathcal{N} = (2,2)$ SYM mainly to fix our notation. Some remarks are made on the continuum limit. In section 3, the results of our Monte Carlo simulation are reported. In section 3.1, we explain our simulation algorithm and related matters. In section 3.2, one-point WT identities are studied. In section 3.3, expectation values of gauge-invariant bi-linear operators of scalar fields are studied. Section 4 is devoted to conclusion. Throughout this paper, the gauge group is assumed to be $\mathrm{SU}(N_c)$ and our simulation has been done only for SU(2).

2. Sugino's lattice formulation of the 2d $\mathcal{N} = (2,2)$ SYM

2.1 Topological field theoretical form of the continuum target theory

This lattice formulation starts with the fact that the (euclidean) action of the 2d $\mathcal{N} = (2,2)$ SYM can be written in the form of the topological field theory [27][2]

[2]The conventional form of the action of the 2d $\mathcal{N} = (2,2)$ SYM, for example, eq. (2.7) of ref. [19], is reproduced by the following substitution

$$\begin{aligned}
&A_\mu \to -ig\sum_a A_\mu^a T^a, \qquad \phi \to -ig\sum_a(\varphi^a + i\phi^a)T^a, \qquad \overline{\phi} \to -ig\sum_a(\varphi^a - i\phi^a)T^a, \\
&\psi_0 \to -ig\sum_a(-i\psi_1^a + i\psi_2^a - \overline{\psi}_1^a + \overline{\psi}_2^a)T^a/2, \qquad \psi_1 \to -ig\sum_a(\psi_1^a - \psi_2^a + i\overline{\psi}_1^a - i\overline{\psi}_2^a)T^a/2, \\
&\chi \to -ig\sum_a(-\psi_1^a - \psi_2^a - i\overline{\psi}_1^a - i\overline{\psi}_2^a)T^a/2, \qquad \eta \to -ig\sum_a(i\psi_1^a + i\psi_2^a + \overline{\psi}_1^a + \overline{\psi}_2^a)T^a/2,
\end{aligned} \tag{2.1}$$

where T^a are anti-hermitian generators of $\mathrm{SU}(N_c)$ normalized as $\mathrm{tr}\{T^aT^b\} = -(1/2)\delta_{ab}$ and the index a runs from 1 to $N_c^2 - 1$.

$$S_{\text{continuum}} = \frac{1}{g^2}\int d^2x \; \mathrm{tr}\Big\{\frac{1}{4}[\phi,\overline{\phi}]^2 + H^2 - iH\Phi + D_\mu\phi D_\mu\overline{\phi}$$
$$- \frac{1}{4}\eta[\phi,\eta] - \chi[\phi,\chi] + \psi_\mu[\overline{\phi},\psi_\mu] + i\chi Q\Phi + i\psi_\mu D_\mu\eta\Big\}, \quad (2.2)$$

where all fields are SU(N_c) Lie algebra valued and scalar fields ϕ and $\overline{\phi}$ are combinations of two real scalar fields, $\phi = X_2 + iX_3$ and $\overline{\phi} = X_2 - iX_3$, respectively. $\Phi = 2F_{01}$ is the field strength in two dimensions $F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1]$. The covariant derivatives D_μ are defined with respect to the adjoint representation $D_\mu\varphi = \partial_\mu\varphi + i[A_\mu, \varphi]$ for any field φ. The index μ runs over 0 and 1. Note that, in the above convention, the bosonic fields A_μ, ϕ and $\overline{\phi}$ have the mass dimension 1 and the fermionic fields ψ_μ, χ and η have the mass dimension 3/2, because the gauge coupling constant in two dimensions g has the mass dimension 1.

In eq. (2.2), Q is a BRST-like transformation in the topological field theory (that is a particular linear combination of super-transformations in the original SYM theory) and is defined by

$$\begin{aligned}
&QA_\mu = \psi_\mu, && Q\psi_\mu = iD_\mu\phi,\\
&Q\phi = 0, && \\
&Q\chi = H, && QH = [\phi,\chi],\\
&Q\overline{\phi} = \eta, && Q\eta = [\phi,\overline{\phi}].
\end{aligned} \quad (2.3)$$

The salient feature of this transformation is that its square Q^2 is an infinitesimal gauge transformation with the transformation parameter ϕ. Therefore, Q is nilpotent $Q^2 = 0$ when acting on gauge invariant quantities. Moreover, the action can be expressed as a Q-exact form:

$$S_{\text{continuum}} = Q\frac{1}{g^2}\int d^2x \; \mathrm{tr}\left\{\frac{1}{4}\eta[\phi,\overline{\phi}] - i\chi\Phi + \chi H - i\psi_\mu D_\mu\overline{\phi}\right\}. \quad (2.4)$$

In this form, the Q-invariance of the action is manifest. Then the idea[3] is to construct a lattice analogue of the Q transformation such that the nilpotency (up to the lattice gauge transformation) holds. Then adopting a lattice action of the structure of eq. (2.4), Q-invariance can be preserved exactly in lattice theory.

2.2 Lattice formulation

We consider two-dimensional square lattice of the one-dimensional physical extent L,

$$\Lambda = \left\{x \in a\mathbb{Z}^2 \mid 0 \le x_\mu < L\right\}, \quad (2.5)$$

where a denotes the lattice spacing. We define also the one-dimensional extent in a lattice unit $N = L/a$. All fields except the gauge potentials are put on sites and, as is conventional

[3]See ref. [28].

in lattice gauge theory, the gauge field is expressed by the compact link variables $U(x,\mu)$. Periodic boundary conditions on Λ are assumed on all fields.

As a lattice counterpart of the fermionic transformation (2.3), we define ($\hat{\mu}$ implies a unit vector in the μ-direction)

$$\begin{aligned} QU(x,\mu) &= i\psi_\mu(x)U(x,\mu), \\ Q\psi_\mu(x) &= i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U(x,\mu)\phi(x+a\hat{\mu})U(x,\mu)^{-1}\right), \\ Q\phi(x) &= 0, \\ Q\chi(x) &= H(x), \qquad QH(x) = [\phi(x),\chi(x)], \\ Q\overline{\phi}(x) &= \eta(x), \qquad Q\eta(x) = [\phi(x),\overline{\phi}(x)]. \end{aligned} \tag{2.6}$$

It can be confirmed that Q^2 is in fact an infinitesimal lattice gauge transformation with the parameter $\phi(x)$. Thus the nilpotency $Q^2 = 0$ holds on gauge invariant quantities. The lattice action is then defined by an expression analogous to eq. (2.4):

$$S = Qa^2 \sum_{x\in\Lambda} \left(\mathcal{O}_1(x) + \mathcal{O}_2(x) + \mathcal{O}_3(x) + \frac{1}{a^4g^2}\,\mathrm{tr}\,\{\chi(x)H(x)\} \right), \tag{2.7}$$

where

$$\mathcal{O}_1(x) = \frac{1}{a^4g^2}\,\mathrm{tr}\left\{ \frac{1}{4}\eta(x)[\phi(x),\overline{\phi}(x)] \right\}, \tag{2.8}$$

$$\mathcal{O}_2(x) = \frac{1}{a^4g^2}\,\mathrm{tr}\left\{ -i\chi(x)\hat{\Phi}(x) \right\}, \tag{2.9}$$

$$\mathcal{O}_3(x) = \frac{1}{a^4g^2}\,\mathrm{tr}\left\{ i\sum_{\mu=0}^{1}\psi_\mu(x)\left(\overline{\phi}(x) - U(x,\mu)\overline{\phi}(x+a\hat{\mu})U(x,\mu)^{-1}\right) \right\}. \tag{2.10}$$

In the above expression, $\hat{\Phi}(x)$ is a lattice analogue of the field strength and is defined from the plaquette variables

$$U(x,0,1) = U(x,0)U(x+a\hat{0},1)U(x+a\hat{1},0)^{-1}U(x,1)^{-1} \tag{2.11}$$

by

$$\hat{\Phi}(x) = \frac{\Phi(x)}{1 - \frac{1}{\epsilon^2}\,\|1 - U(x,0,1)\|^2} \tag{2.12}$$

with

$$\Phi(x) = -i\left[U(x,0,1) - U(x,0,1)^{-1}\right]. \tag{2.13}$$

Finally, the matrix norm in the above expression is defined by

$$\|A\| = \left[\mathrm{tr}\left\{AA^\dagger\right\}\right]^{1/2} \tag{2.14}$$

and the constant ϵ is chosen as (for $N_c = 2$)

$$0 < \epsilon < 2\sqrt{2}. \tag{2.15}$$

(The meaning of the denominator of eq. (2.12) will be explained shortly.) From the Q-exact form (2.7) and the nilpotency of Q, the lattice action is manifestly invariant under the Q-transformation (2.6).[4]

After the operation of Q, the lattice action becomes

$$S = a^2 \sum_{x\in\Lambda} \left(\sum_{i=1}^{3} \mathcal{L}_{\mathrm{B}i}(x) + \sum_{i=1}^{6} \mathcal{L}_{\mathrm{F}i}(x) + \frac{1}{a^4 g^2} \operatorname{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{\mathrm{TL}}(x) \right\}^2 \right), \tag{2.16}$$

where we have noted that only the *traceless part* of $\hat{\Phi}(x)$,

$$\hat{\Phi}_{\mathrm{TL}}(x) = \hat{\Phi}(x) - \frac{1}{N_c} \operatorname{tr} \left\{ \hat{\Phi}(x) \right\} \mathbb{1}, \tag{2.17}$$

appears in the action, because the auxiliary field $H(x)$ is traceless [2]. Each term of the action density is given by

$$\mathcal{L}_{\mathrm{B}1}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ \frac{1}{4} [\phi(x), \overline{\phi}(x)]^2 \right\}, \tag{2.18}$$

$$\mathcal{L}_{\mathrm{B}2}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ \frac{1}{4} \hat{\Phi}_{\mathrm{TL}}(x)^2 \right\}, \tag{2.19}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{B}3}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \Bigg\{ \sum_{\mu=0}^{1} & \left(\phi(x) - U(x,\mu) \phi(x + a\hat{\mu}) U(x,\mu)^{-1} \right) \\ & \times \left(\overline{\phi}(x) - U(x,\mu) \overline{\phi}(x + a\hat{\mu}) U(x,\mu)^{-1} \right) \Bigg\}, \end{aligned} \tag{2.20}$$

and

$$\mathcal{L}_{\mathrm{F}1}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ -\frac{1}{4} \eta(x) [\phi(x), \eta(x)] \right\}, \tag{2.21}$$

$$\mathcal{L}_{\mathrm{F}2}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ -\chi(x) [\phi(x), \chi(x)] \right\}, \tag{2.22}$$

$$\mathcal{L}_{\mathrm{F}3}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ -\psi_0(x) \psi_0(x) \left(\overline{\phi}(x) + U(x,0) \overline{\phi}(x + a\hat{0}) U(x,0)^{-1} \right) \right\}, \tag{2.23}$$

$$\mathcal{L}_{\mathrm{F}4}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ -\psi_1(x) \psi_1(x) \left(\overline{\phi}(x) + U(x,1) \overline{\phi}(x + a\hat{1}) U(x,1)^{-1} \right) \right\}, \tag{2.24}$$

$$\mathcal{L}_{\mathrm{F}5}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ i \chi(x) Q \hat{\Phi}(x) \right\}, \tag{2.25}$$

$$\mathcal{L}_{\mathrm{F}6}(x) = \frac{1}{a^4 g^2} \operatorname{tr} \left\{ -i \sum_{\mu=0}^{1} \psi_\mu(x) \left(\eta(x) - U(x,\mu) \eta(x + a\hat{\mu}) U(x,\mu)^{-1} \right) \right\}. \tag{2.26}$$

It is important to keep in mind that all lattice fields in the above expressions are dimensionless. For comparison of correlation functions with the continuum theory (2.2), we have

[4] Another important property of the present lattice formulation is a manifestly preserved global $\mathrm{U}(1)_R$ symmetry [15, 1, 2].

to rescale all lattice fields (and Q) by appropriate factors of $1/a$ according to their mass dimension.

With the lattice action (2.16), the expectation value of an operator $\mathcal{O}$ is defined as usual

$$\langle\!\langle \mathcal{O} \rangle\!\rangle = \frac{\displaystyle\int \mathrm{d}\mu\, \mathcal{O}\, e^{-S}}{\displaystyle\int \mathrm{d}\mu\, e^{-S}}, \tag{2.27}$$

where the integration measure is defined by (writing $\phi(x) = X_2(x) + iX_3(x)$ and $\overline{\phi}(x) = X_2(x) - iX_3(x)$)

$$\mathrm{d}\mu = \prod_{x\in\Lambda} \left(\prod_{\mu=0}^{1} \mathrm{d}U(x,\mu) \right) \prod_{a=1}^{N_c^2-1} \mathrm{d}X_2^a(x)\, \mathrm{d}X_3^a(x)\, \mathrm{d}H^a(x) \left(\prod_{\mu=0}^{1} \mathrm{d}\psi_\mu^a(x) \right) \mathrm{d}\chi^a(x)\, \mathrm{d}\eta^a(x) \tag{2.28}$$

in terms of color components of fields, $\varphi(x) = -i\sum_{a=1}^{N_c^2-1} \varphi^a(x) T^a$, where T^a are anti-hermitian generators of $\mathrm{SU}(N_c)$ (normalized as $\mathrm{tr}\{T^a T^b\} = -(1/2)\delta_{ab}$). $\mathrm{d}U(x,\mu)$ is the standard Haar measure. Note that the integration over the auxiliary field $H(x)$ is gaussian and can be done readily. The invariance of this measure under the Q-transformation is noted in the last reference of ref. [15].

The denominator of eq. (2.12) needs an explanation. Without that factor, the lattice action for the gauge field is the "double-winding plaquette type" [29] and the action possesses many degenerate minima which have no continuum counterpart. Due to the denominator of eq. (2.12), the action (2.16) diverges as $\|1 - U(x,0,1)\| \to \epsilon$ at a certain site x. Precisely speaking, the above construction of the action is applied only for configurations with

$$\|1 - U(x,0,1)\| < \epsilon, \qquad \text{for } \forall x \in \Lambda, \tag{2.29}$$

and, otherwise, i.e., if there exists $x \in \Lambda$ such that $\|1 - U(x,0,1)\| \geq \epsilon$, we set

$$S = +\infty. \tag{2.30}$$

In this way, the domain of functional integral (2.27) is effectively restricted to the space specified by the condition (2.29)[5] and, by setting the parameter ϵ in the range (2.15), it can be shown that the unique physical minimum of the action (up to gauge transformations) is singled out. This procedure to solve the problem of degenerate minima does not break the Q-symmetry. See ref. [2] for careful discussion on these points.

With the above construction, one fermionic symmetry Q is manifestly preserved on the lattice. The price to pay is that the pfaffian, resulting from the integration over fermionic fields, is generally complex[6] and this could be disadvantage in Monte Carlo simulation. We will see below that, however, this point appears to be not problematic, at least with the parameters in our numerical study.

[5] This is the so-called admissibility condition considered in a different context [30].

[6] In the target continuum theory, the corresponding pfaffian is real and positive semi-definite.

2.3 Continuum limit

In the present two-dimensional super-renormalizable system, all dimension-ful quantities can be measured by taking the gauge coupling constant g, which has the mass dimension 1, as a unit (in this sense, g is analogous to the Λ-parameter in QCD).[7] The continuum limit is defined by the limit $a \to 0$, while g and L, the physical extent of the two-dimensional space, are kept fixed. In refs. [15, 1], the restoration of *full* set of supersymmetry in this continuum limit was argued on the basis of the loop expansion and power counting.[8] More precisely, this argument shows that the 1PI effective action for *elementary fields* is supersymmetric in the continuum limit. Note that the argument of refs. [15, 1] says nothing about possible supersymmetry breaking in correlation functions that contain *composite fields*.

Now, in numerical study, it is convenient to define the dimensionless gauge coupling constant by

$$\frac{\beta}{2N_c} = \frac{1}{a^2 g^2}, \tag{2.31}$$

that is simply the over-all common coefficient of the lattice action (2.16). Clearly, β goes infinity in the continuum limit. In terms of β, the lattice spacing in a unit of the gauge coupling constant g is given by

$$a = \sqrt{\frac{2N_c}{\beta}}\frac{1}{g}, \tag{2.32}$$

and, correspondingly, the one-dimensional physical extent of the lattice is

$$L = aN = \sqrt{\frac{2N_c}{\beta}}N\frac{1}{g}, \tag{2.33}$$

where N is the one-dimensional size in a lattice unit.

As already noted, all fields on the lattice must be rescaled by appropriate factors of $1/a$, for comparison with the continuum theory (2.2). All bosonic fields in the continuum theory (except the auxiliary field), which have the mass dimension 1, are related to the lattice fields by

$$\varphi_{\text{continuum}}(x) = \frac{1}{a}\varphi(x) = \sqrt{\frac{\beta}{2N_c}}g\varphi(x) \tag{2.34}$$

and the correlation functions are measured in a unit of g. Similarly, fermionic fields are related as

$$\psi_{\text{continuum}}(x) = \frac{1}{a^{3/2}}\psi(x) = \left(\frac{\beta}{2N_c}\right)^{3/2} g^{3/2}\psi(x). \tag{2.35}$$

Note that, in the continuum limit, these rescalings *amplify* the correlation functions on the lattice.

[7] Recall also that, in the present system, there is no non-trivial coupling constant renormalization nor wave function renormalization. Only mass terms of bosonic fields may be renormalized (ignoring gauge symmetry and supersymmetry).

[8] Strictly speaking, this argument as it stands holds for the limit $a \to 0$ with the fixed number of lattice points $N = L/a$ (thus the physical lattice size goes to zero $L = aN \to 0$). The argument, however, can slightly be modified to show a restoration of supersymmetry in the present (L fixed) continuum limit, to all orders of the loop expansion.

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3. Monte Carlo study

3.1 Algorithm, simulation code and statistics

In supersymmetric theories, the quantum effect of fermions is vital and the quenched approximation is almost meaningless. Even in the present two-dimensional system, a treatment of dynamical fermions can be non-trivial and costly. The 2d $\mathcal{N} = (2, 2)$ SYM can be obtained by dimensional reduction of the 4d $\mathcal{N} = 1$ SYM in which the fermion field is a *Majorana* spinor instead of Dirac. Thus the pfaffian of the Dirac operator, instead of the determinant, naturally appears. In a sense, we have to treat an $N_f = 1/2$ system. To compute the pseudo-fermion force in the hybrid Monte Carlo algorithm, one then has to implement the fourth-root of $D^\dagger D$, where D is a lattice Dirac operator. Moreover, this fermion must be massless (at least in the continuum limit). Thus the numerical simulation of the 4d SYM is quite demanding.

In two dimensions, on the other hand, it should be relatively easy to accumulate high statistics compared to four dimensions. Taking these things into consideration, here we adopt a (somewhat brute force) re-weighting method.[9] That is, we prepare configurations with the statistical weight e^{-S_B}, where S_B is the lattice action (2.16) with all fermion fields are removed. This is a quenched ensemble. Writing the expectation value in this purely bosonic system by[10]

$$\langle \mathcal{O} \rangle = \frac{\displaystyle\int \mathrm{d}\mu_\mathrm{B}\, \mathcal{O}\, e^{-S_\mathrm{B}}}{\displaystyle\int \mathrm{d}\mu_\mathrm{B}\, e^{-S_\mathrm{B}}}, \tag{3.1}$$

the true expectation value is evaluated by re-weighting configurations by the factor of pfaffian[11]

$$\langle\!\langle \mathcal{O} \rangle\!\rangle = \frac{\langle \mathcal{O}\, \mathrm{Pf}\{D\} \rangle}{\langle \mathrm{Pf}\{D\} \rangle}, \tag{3.2}$$

where D is the lattice Dirac operator appeared in the action (2.16). Mathematically, this definition is equivalent to the original one (2.27). Practically, however, we have only a limited number of configurations and there may exist the overlap problem. That is, distribution of configurations favored by the the quenched weight e^{-S_B} may not have a sufficient overlap with that of configurations really important in the original system. So we need many configurations to reproduce the true expectation values in the original unquenched system.

We developed a C++ code of the hybrid Monte Carlo simulation with the action S_B by using a library due to Massimo Di Pierro, the FermiQCD/MDP [33]. For each con-

[9] Hidenori Fukaya suggested this method to me. For application in two dimensions, see, for example, ref. [31].

[10] When the operator $\mathcal{O}$ contains fermionic fields, they are contracted by fermion propagators in the presence of bosonic fields.

[11] Note that, in this method, $\langle\!\langle \mathcal{O} \rangle\!\rangle$ is evaluated by a *ratio* of two averages over an ensemble. This means that $\langle\!\langle \mathcal{O} \rangle\!\rangle$ is not the primary quantity [32] and care is needed to estimate the statistical error in $\langle\!\langle \mathcal{O} \rangle\!\rangle$. We used the jackknife analysis to estimate the average and the statistical error for $\langle\!\langle \mathcal{O} \rangle\!\rangle$. I would like to thank Issaku Kanamori for clarifying discussion on this point.

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N	8	7	6	5	4
β	16.0	12.25	9.0	6.25	4.0
number of configs.	1,000	10,000	10,000	10,000	10,000
ag	0.5	0.571428	0.666666	0.8	1.0

Table 1: Parameters in our Monte Carlo study. This sequence corresponds to the fixed physical lattice size $Lg = 4.0$.

figuration, we compute the inverse (i.e., the fermion propagator) and the determinant of the lattice Dirac operator D by using the LU decomposition. *We do not introduce any (supersymmetry breaking) mass terms of fermions and bosons.*

We carried out simulations with the parameters in table 1. The sequence, according to eqs. (2.32) and (2.33), corresponds to a fixed physical lattice size $Lg = 4.0$ and the lattice spacings $ag = 1.0$, 0.8, 0.666, 0.571 and 0.5, respectively. For each value of β, we stored 1,000–10,000 independent configurations extracted from 10^6 trajectories of the molecular dynamics. The constant ϵ in eq. (2.12) is kept fixed at $\epsilon = 2.6$.[12]

Expressing the determinant of the Dirac operator in the form

$$\det\{D\} = re^{i\theta}, \qquad -\pi < \theta \leq \pi, \tag{3.3}$$

(generally the determinant is complex due to lattice artifacts in the present formulation) we evaluate the pfaffian by

$$\mathrm{Pf}\{D\} = \sqrt{r}e^{i\theta/2}, \tag{3.4}$$

because $(\mathrm{Pf}\{D\})^2 = \det\{D\}$. This prescription, however, may give a wrong *sign* for the pfaffian. For example, if $\mathrm{Pf}\{D\} = \sqrt{r}e^{(2/3)\pi i}$, we have $\theta = -2\pi/3$ and the prescription (3.4) gives $\mathrm{Pf}\{D\} = \sqrt{r}e^{-(1/3)\pi i} = -\sqrt{r}e^{(2/3)\pi i}$ which is wrong in sign. The prescription (3.4) gives the correct sign of the pfaffian, provided that $-\pi/2 < \mathrm{Arg}(\mathrm{Pf}\{D\}) \leq \pi/2$ (and otherwise the prescription gives a wrong sign). Although this is expected to be the case for large β (i.e., when close to the continuum), to determine the true sign of the pfaffian, we have to compute the pfaffian itself in some direct way. This is quite time-consuming[13] and we do not adopt this method in this paper. Instead, to have an idea how the prescription (3.4) works in practice, we measured the distribution of the pfaffians over a subset of our ensemble in table 1. The behavior in the figure 1 clearly accords with our expectation. For large β (i.e., close to the continuum), the distribution gathers around the positive side of the real axis and the condition $-\pi/2 < \mathrm{Arg}(\mathrm{Pf}\{D\}) \leq \pi/2$ is fulfilled. Even for the smallest β in our simulation, $\beta = 4.0$, the distribution is significantly biased on the

[12]We observed a tendency such that the autocorrelation time becomes shorter for smaller ϵ. Thus small ϵ would be favorable from a viewpoint to accumulate a large number of configurations. On the other hand, it appears that smaller ϵ implies smaller fluctuation of distribution of configurations and might be disadvantageous from a viewpoint of the overlap problem. We did not systematically investigate this problem of an optimal ϵ. Our present ϵ is rather large in view of eq. (2.15).

[13]It can be seen that the algorithm for the pfaffian (appearing, for example, in ref. [5]) is an $O(n^4)$-process for a $2n \times 2n$ matrix, while the LU decomposition has an $O(n^3)$-process algorithm.

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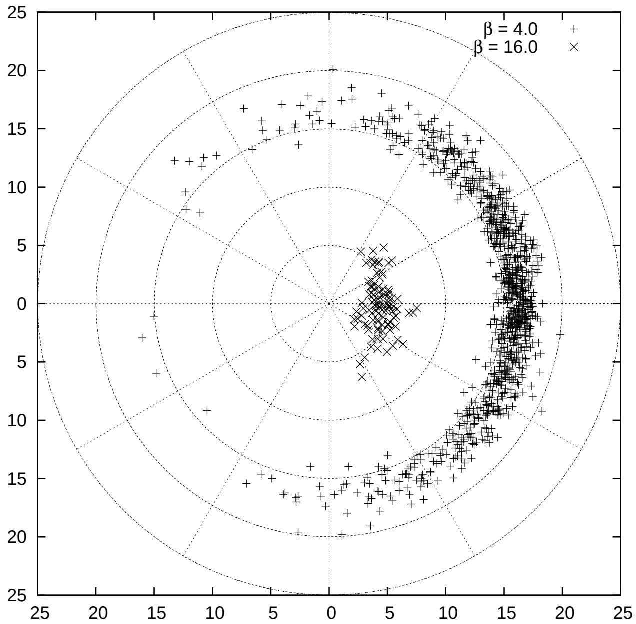

Figure 1: The distribution of pfaffians in a subset of the quenched ensemble used in our simulation. The phase, $\mathrm{Arg}(\mathrm{Pf}\{D\})$, and the modulus in logarithm, $\log_{10}(10^{16}|\,\mathrm{Pf}\{D\}|)$, are plotted in the polar coordinate. The number of samples is 1,000 and 100 for $\beta = 4.0$ and $\beta = 16.0$, respectively.

side of the positive real axis. Thus the systematic error introduced by the wrong-sign determination due to the prescription (3.4) would be negligible compared to the statistical error.

We consider also the quenched approximation, i.e.,

$$\langle\!\langle \mathcal{O} \rangle\!\rangle_{\text{quenched}} = \frac{\langle \mathcal{O} \rangle}{\langle 1 \rangle}. \tag{3.5}$$

This provides a useful standard with which one can observe the extent of the quantum effect of dynamical fermions.

3.2 One-point supersymmetric WT identities

First, we consider supersymmetric one-point WT identities implied by the exact Q-invariance of the lattice action. These are of the form $\langle\!\langle Q(\text{something}) \rangle\!\rangle$ and identically vanish because of Q-invariance of the action and the integration measure. These should hold for any lattice parameter, if the integration (especially that over fermionic fields) is properly performed. Thus, from their validity in numerical simulation, we can confirm the correctness of our code/algorithm. In particular, we can observe whether the re-weighting method works or not.

Since the lattice action (2.7) is Q-exact, we have $\langle\!\langle S \rangle\!\rangle = 0$, or, in terms of the action density,

$$\sum_{i=1}^{3} \langle\!\langle \mathcal{L}_{\mathrm{B}i}(x) \rangle\!\rangle + \sum_{i=1}^{6} \langle\!\langle \mathcal{L}_{\mathrm{F}i}(x) \rangle\!\rangle + \frac{1}{a^4 g^2} \left\langle\!\!\left\langle \mathrm{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{\mathrm{TL}}(x) \right\}^2 \right\rangle\!\!\right\rangle = 0. \tag{3.6}$$

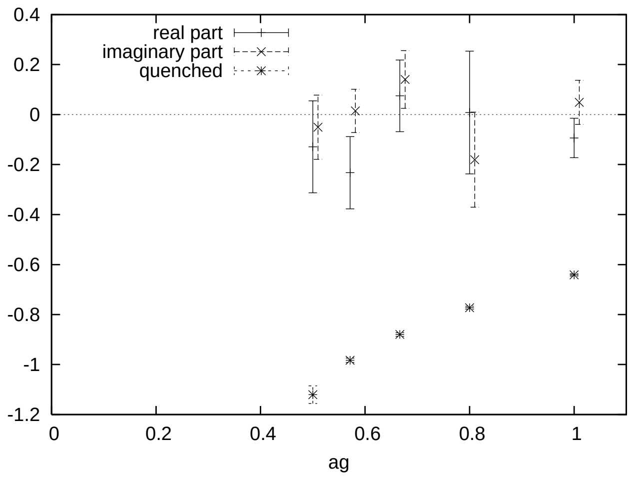

Figure 2: Expectation values of $\sum_{i=1}^{3} \mathcal{L}_{\mathrm{B}i}(x) - (3/2)(N_c^2-1)a^{-2}$.

One may further simplify this relation. The second term is the expectation value of the action density of the fermionic fields. Since the action is bi-linear in fermionic fields, we have $\sum_{i=1}^{6} \langle\!\langle \mathcal{L}_{\mathrm{F}i}(x)\rangle\!\rangle = -2(N_c^2-1)a^{-2}$ (the coefficient 2 is $(1/2)\times 4$, where 1/2 reflects the Majorana nature of the system and 4 is the number of fermion species). Similarly, the auxiliary field $H(x)$ can be integrated out and the last term becomes $(1/2)(N_c^2-1)a^{-2}$ after integration. Thus

$$\sum_{i=1}^{3} \langle\!\langle \mathcal{L}_{\mathrm{B}i}(x)\rangle\!\rangle - \frac{3}{2}(N_c^2-1)\frac{1}{a^2} = 0. \tag{3.7}$$

In figure 2, we plotted the left-hand side of this relation (in a unit of g^2) as a function of the lattice spacing ag. The real part is consistent with the expected identity (3.7) within 1σ for all values of ag, except $ag = 0.571$ that is 1.5σ away. This agreement strongly indicates the correctness of our code/algorithm. The average of the imaginary part is consistent with zero, as it should be, although its fluctuation is comparable to that of the real part.

What is intriguing with figure 2 is that one can see clear distinction between the re-weighted average (3.2) and the quenched average (3.5). This illustrates that the re-weighting method works very well and the effect of dynamical fermions is properly included (at least for the present quantity). A perturbative argument shows that each term of the action density behaves as

$$\langle\!\langle \mathcal{L}_{\mathrm{B}2}(x)\rangle\!\rangle \sim \frac{1}{2}(N_c^2-1)\frac{1}{a^2}, \qquad \langle\!\langle \mathcal{L}_{\mathrm{B}3}(x)\rangle\!\rangle \sim (N_c^2-1)\frac{1}{a^2}, \tag{3.8}$$

in $a \to 0$ because of one-loop diagrams and $\langle\!\langle \mathcal{L}_{\mathrm{B}1}(x)\rangle\!\rangle$ starts with a two-loop diagram which behaves as $\sim (\ln(a/L))^2 g^2$. The leading $O(a^{-2})$ singularities are thus cancelled out in the sum $\sum_{i=1}^{3} \langle\!\langle \mathcal{L}_{\mathrm{B}i}(x)\rangle\!\rangle - (3/2)(N_c^2-1)a^{-2}$ and this leaves a function of the form $f(a/L, Lg)g^2$. This function identically vanishes if supersymmetry holds, but it is a non-trivial function in the quenched approximation. What is shown in the figure 2 with "quenched" is this function.[14]

[14] The lattice perturbation theory is not useful to evaluate this function even for $ag \to 0$. In the loop

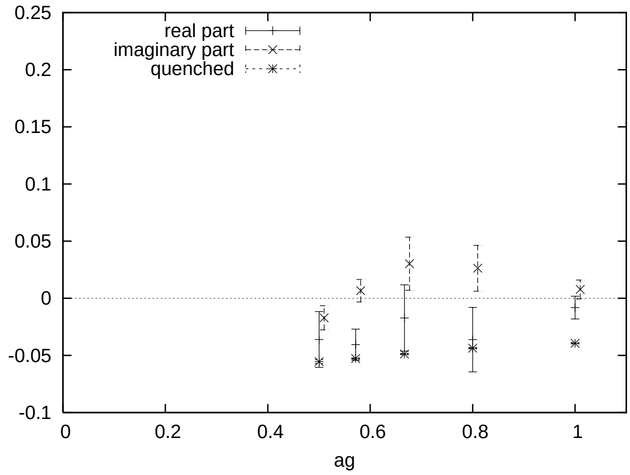

Figure 3: Expectation values of $\mathcal{L}_{\mathrm{B1}}(x) + \mathcal{L}_{\mathrm{F1}}(x)$.

The identity (3.6) can be divided into several pieces, each of which should hold separately. The first one is

$$\langle\!\langle Q\mathcal{O}_1(x)\rangle\!\rangle = \langle\!\langle \mathcal{L}_{\mathrm{B1}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F1}}(x)\rangle\!\rangle = 0, \tag{3.9}$$

and the left-hand side is plotted in figure 3. The relation is confirmed within 1.5σ expect the case $ag = 0.571$. Note the difference in scale of vertical axis compared to figure 2. Although the results with a quenched ensemble are certainly inconsistent with the supersymmetric relation (3.9), we do not see clear separation between the re-weighted average and the quenched average. This seems to be related to the fact that diagrams that contribute to $\langle\!\langle \mathcal{L}_{\mathrm{B1}}(x)\rangle\!\rangle$ and $\langle\!\langle \mathcal{L}_{\mathrm{F1}}(x)\rangle\!\rangle$ and contain virtual fermion loops start with three loops, a rather higher order.

Another piece of eq. (3.6) is

$$\langle\!\langle Q\mathcal{O}_2(x)\rangle\!\rangle = \frac{1}{a^4 g^2}\left\langle\!\!\left\langle \mathrm{tr}\left\{-iH(x)\hat{\Phi}_{\mathrm{TL}}(x)\right\}\right\rangle\!\!\right\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F5}}(x)\rangle\!\rangle = 0. \tag{3.10}$$

Under the gaussian integration, the auxiliary field can be replaced by $H(x) = \frac{1}{2}i\hat{\Phi}_{\mathrm{TL}}(x)$ and the above becomes

$$2\,\langle\!\langle \mathcal{L}_{\mathrm{B2}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F5}}(x)\rangle\!\rangle = 0. \tag{3.11}$$

In figure 4, the left-hand side of this relation is plotted. The global feature is similar to that of figure 2 and the relation is reproduced within almost 1σ.

The situation is again similar with the last piece of the relation (3.6):

$$\langle\!\langle Q\mathcal{O}_3(x)\rangle\!\rangle = \langle\!\langle \mathcal{L}_{\mathrm{B3}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F3}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F4}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F6}}(x)\rangle\!\rangle = 0, \tag{3.12}$$

whose left-hand side is plotted in figure 5.

expansion, we may have terms of the form, say, $(\ln(a/L))^{\ell}(Lg)^{2(\ell-2)}g^2$ at ℓ-loop level. All this type of terms equally contribute to the function f in the present continuum limit, in which Lg is fixed (recall that $Lg = 4.0$ in our simulation).

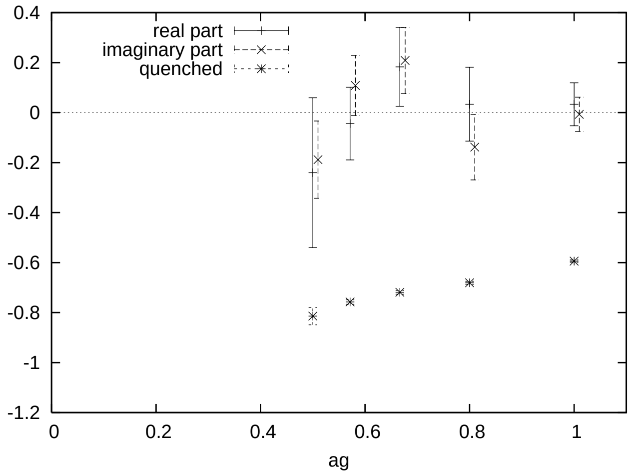

Figure 4: Expectation values of $2\mathcal{L}_{\mathrm{B2}}(x) + \mathcal{L}_{\mathrm{F5}}(x)$.

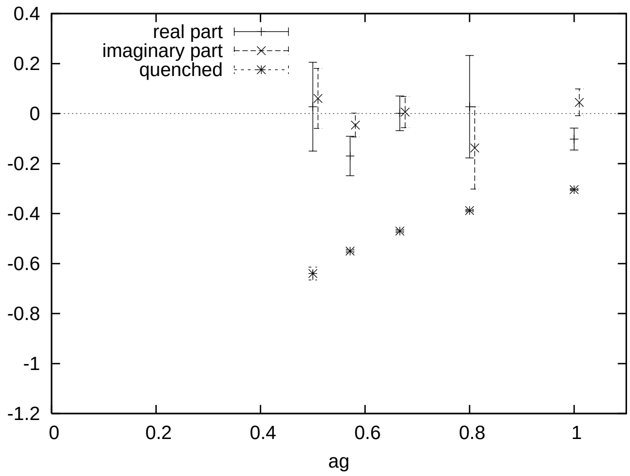

Figure 5: Expectation values of $\mathcal{L}_{\mathrm{B3}}(x) + \mathcal{L}_{\mathrm{F3}}(x) + \mathcal{L}_{\mathrm{F4}}(x) + \mathcal{L}_{\mathrm{F6}}(x)$.

So far, we have observed WT identities implied by the exact Q-symmetry of the lattice action. The continuum theory (2.2), on the other hand, is invariant under also other fermionic transformations, Q_{01}, Q_0 and Q_1. In the lattice framework, the invariance under these transformations is expected to be restored only in the continuum limit. The fermionic transformation Q_{01} is given by

$$\begin{aligned}
Q_{01}A_\mu &= -\epsilon_{\mu\nu}\psi_\mu, & Q_{01}\psi_\mu &= i\epsilon_{\mu\nu}D_\nu\phi, \\
Q_{01}\phi &= 0, \\
Q_{01}\eta &= 2H, & Q_{01}H &= \frac{1}{2}[\phi,\eta], \\
Q_{01}\overline{\phi} &= -2\chi, & Q_{01}\chi &= -\frac{1}{2}[\phi,\overline{\phi}],
\end{aligned} \tag{3.13}$$

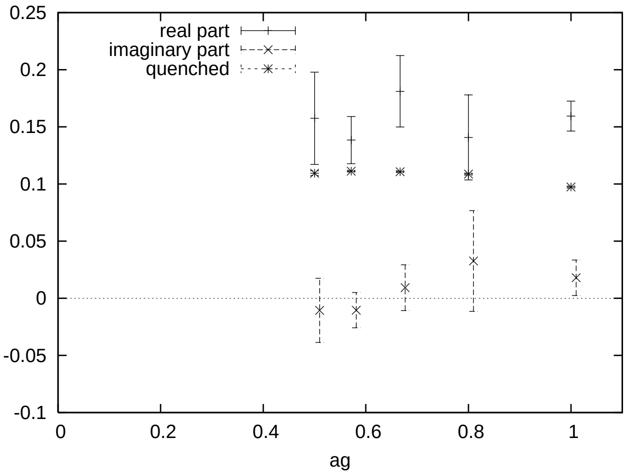

Figure 6: Expectation values of $\mathcal{L}_{\mathrm{B1}}(x) + \mathcal{L}_{\mathrm{F2}}(x)$.

that can be obtained by following substitutions in the Q-transformation (2.3)

$$\frac{1}{2}\eta \to -\chi, \qquad \chi \to \frac{1}{2}\eta, \qquad \psi_\mu \to -\epsilon_{\mu\nu}\psi_\nu, \tag{3.14}$$

where $\epsilon_{01} = -\epsilon_{10} = 1$. Since the action (2.2) is invariant under these substitutions, the invariance of the continuum action under eq. (3.13) is obvious. Associated with this Q_{01}-invariance, in the supersymmetric continuum theory, we have

$$\begin{aligned}\left\langle\!\left\langle Q_{01}\frac{1}{g^2}\,\mathrm{tr}\left\{-\frac{1}{2}\chi[\phi,\bar{\phi}]\right\}\right\rangle\!\right\rangle_{\mathrm{continuum}} \\ = \frac{1}{g^2}\left\langle\!\left\langle \mathrm{tr}\left\{\frac{1}{4}[\phi,\bar{\phi}]^2\right\}\right\rangle\!\right\rangle_{\mathrm{continuum}} + \frac{1}{g^2}\left\langle\!\left\langle \mathrm{tr}\left\{-\chi[\phi,\chi]\right\}\right\rangle\!\right\rangle_{\mathrm{continuum}} = 0.\end{aligned} \tag{3.15}$$

Thus, corresponding to this relation, one might expect

$$\langle\!\langle \mathcal{L}_{\mathrm{B1}}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\mathrm{F2}}(x)\rangle\!\rangle \to 0 \tag{3.16}$$

holds in the continuum limit $a \to 0$.

In figure 6, we plotted the left-hand side of eq. (3.16). It appears that the average approaches a non-zero number around 0.15, instead of zero (the imaginary part is consistent with zero, as it should be). This does *not* contradict with the supersymmetry restoration. As already noted, the argument [15] for a restoration of supersymmetry in the continuum limit is not applied to correlation functions containing composite operators. In particular, there is no general guarantee that the bare WT identity (3.16) holds in the continuum limit.

We note that if supersymmetry in the 1PI effective action is restored in the continuum limit, it is UV finite, that is, all 1PI diagrams with external lines of elementary fields are UV finite. Power counting (taking gauge invariance into account) shows that only scalar mass terms suffer from superficial UV divergence. Scalar mass terms are, however, inconsistent with supersymmetry. So if the 1PI effective action is supersymmetric, it is UV finite.[15] On the other hand, composite operators $\mathcal{L}_{\rm B1}(x)$ and $\mathcal{L}_{\rm F2}(x)$ induce logarithmic UV divergence at two-loop level. If supersymmetry of the 1PI effective action is restored, this two-loop level divergence, caused by the presence of composite operators, is the only source of UV divergence in $\langle\!\langle \mathcal{L}_{\rm B1}(x)\rangle\!\rangle$ and $\langle\!\langle \mathcal{L}_{\rm F2}(x)\rangle\!\rangle$. Moreover, that remaining two-loop level divergence is cancelled out in the sum $\langle\!\langle \mathcal{L}_{\rm B1}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\rm F2}(x)\rangle\!\rangle$. This argument shows that, if supersymmetry in the 1PI effective action restores, the dependence of $\langle\!\langle \mathcal{L}_{\rm B1}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\rm F2}(x)\rangle\!\rangle$ on ag decreases as $ag \to 0$, i.e., it approaches a constant (but not necessarily zero). The behavior in figure 6 is consistent with this picture based on a restoration of supersymmetry.

What is not completely clear to us is that even the quenched average seems to have the same behavior. Actually, within almost 1σ the re-weighted average and the quenched average are degenerate. So, although figure 6 is consistent with a scenario of a supersymmetry restoration, we cannot conclude the restoration of supersymmetry from the above result.

The continuum action is invariant under also

$$\begin{aligned} Q_0 A_0 &= \frac{1}{2}\eta, & Q_0\eta &= -2iD_0\overline{\phi}, \\ Q_0 A_1 &= -\chi, & Q_0\chi &= iD_1\overline{\phi}, \\ Q_0\overline{\phi} &= 0, & & \\ Q_0\psi_1 &= -H, & Q_0 H &= [\overline{\phi}, \psi_1], \\ Q_0\phi &= -2\psi_0, & Q_0\psi_0 &= \frac{1}{2}[\overline{\phi}, \phi], \end{aligned} \tag{3.17}$$

that can be obtained by the substitutions in eq. (2.3)

$$\frac{1}{2}\eta \to \psi_0, \qquad \chi \to -\psi_1, \qquad \psi_0 \to \frac{1}{2}\eta, \qquad \psi_1 \to -\chi, \qquad \phi \to -\overline{\phi}, \qquad \overline{\phi} \to -\phi. \tag{3.18}$$

Corresponding to this symmetry, we have

$$\begin{aligned} &\left\langle\!\!\left\langle Q_0 \frac{1}{g^2}\,\mathrm{tr}\left\{-\frac{1}{2}\psi_0[\phi, \overline{\phi}]\right\}\right\rangle\!\!\right\rangle_{\rm continuum} \\ &\qquad = \frac{1}{g^2}\left\langle\!\!\left\langle \mathrm{tr}\left\{\frac{1}{4}[\phi, \overline{\phi}]^2\right\}\right\rangle\!\!\right\rangle_{\rm continuum} + \frac{1}{g^2}\left\langle\!\left\langle \mathrm{tr}\left\{-\psi_0[\psi_0, \overline{\phi}]\right\}\right\rangle\!\right\rangle_{\rm continuum} = 0 \end{aligned} \tag{3.19}$$

and one might expect

$$\langle\!\langle \mathcal{L}_{\rm B1}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\rm F3}(x)\rangle\!\rangle \to 0, \tag{3.20}$$

in the continuum limit $a \to 0$. The result (figure 7) is similar to the previous one. The average seems to approach a non-zero number around 0.05 and we may repeat the above argument.

Another fermionic symmetry Q_1 is obtained by further exchange $\psi_0 \leftrightarrow \psi_1$ in eq. (3.18). Corresponding to this, one might expect

$$\langle\!\langle \mathcal{L}_{\rm B1}(x)\rangle\!\rangle + \langle\!\langle \mathcal{L}_{\rm F4}(x)\rangle\!\rangle \to 0. \tag{3.21}$$

The result of numerical study is plotted in figure 8. The result is very similar to that of figure 7.

[15] The converse is not true. The UV finiteness of the effective action does not imply supersymmetry, as finite scalar mass terms are allowed for the former.

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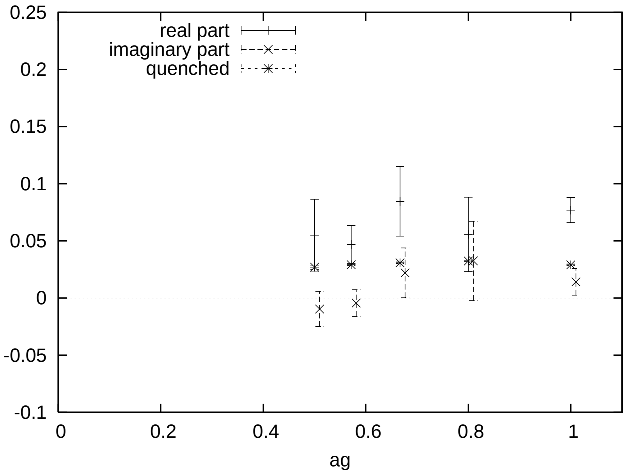

Figure 7: Expectation values of $\mathcal{L}_{\mathrm{B1}}(x) + \mathcal{L}_{\mathrm{F3}}(x)$.

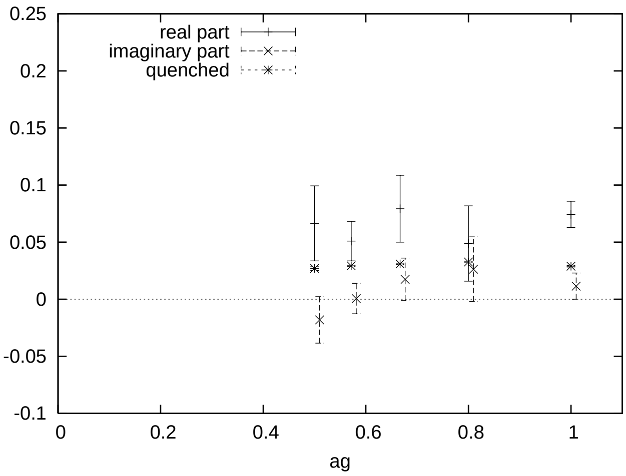

Figure 8: Expectation values of $\mathcal{L}_{\mathrm{B1}}(x) + \mathcal{L}_{\mathrm{F4}}(x)$.

3.3 Expectation value of scalar bi-linear operators

To illustrate possible use of lattice simulation of the present kind, in this section, we consider expectation values of gauge-invariant bi-linear operators of scalar fields, $a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}$ and $a^{-2}\,\mathrm{tr}\{\phi(x)\phi(x)\}$ (the factor a^{-2} is multiplied for the rescaling (2.34)). The classical action of the 2d $\mathcal{N} = (2,2)$ SYM vanishes identically for all configurations of constant scalar fields such that $[\phi, \overline{\phi}] = 0$ (other fields are set to zero). These are so-called flat-directions and classical vacua are infinitely degenerate. Moreover, this degeneracy is not lifted by radiative corrections to all order of perturbative theory. Thus, the expectation values of scalar fields in quantum theory are of interest and, if Monte Carlo simulation is useful, a prediction on these expectation values should be feasible.

First, we consider $a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}$. This operator is invariant under the global $\mathrm{U}(1)_R$ transformation, which acts on scalar fields as $\phi(x) \to e^{2i\alpha}\phi(x)$ and $\overline{\phi}(x) \to e^{-2i\alpha}\overline{\phi}(x)$. The continuum limit of this quantity itself is meaningless, because it is a bare quantity

$L/a = N$	8	7	6	5	4
$c(a/L)$	0.379295	0.357928	0.333234	0.304	0.268229

Table 2: The counter constant $c(a/L)$ given by eq. (3.23).

and suffers from UV divergence. It should be renormalized. A power counting argument shows that the superficial UV divergence comes from the simplest one-loop diagram and the divergence is logarithmic $\sim \ln(a/L)g^2$. If supersymmetry of the 1PI effective action is restored in the continuum limit, as we assume at the moment, this one-loop divergence is the only source of UV divergence of $\langle\langle a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}\rangle\rangle$ (recall the argument below eq. (3.16)).

So we define the renormalized operator (the normal product)

$$\mathcal{N}[a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}] \equiv a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\} - (N_c^2-1)c(a/L)g^2 \tag{3.22}$$

by subtracting a c-number, the value of the one-loop diagram. This subtraction must remove all the UV divergence of the composite operator. This simplicity is a special property of the present two-dimensional (supersymmetric) theory.

The coefficient $c(a/L)$ of the counter constant is given by a simple scalar one-loop diagram and, on a finite size lattice, it is

$$c(a/L=1/N) = \frac{1}{2N^2}\sum_{n_0=0}^{N-1}\sum_{n_1=0}^{N-1}\frac{1}{\sum_{\mu=0}^{1}\left(1-\cos\frac{2\pi}{N}n_\mu\right)}. \tag{3.23}$$

As possible prescription for the zero mode, we do not include $(n_0,n_1)=(0,0)$ in the sum. Values of this counter constant are listed in table 2 for the cases in our simulation.

The result of our Monte Carlo simulation is figure 9.[16] First of all, we see clear separation between the re-weighted average and the quenched one. The difference is thus due to the effect of dynamical fermions. This effect uplifts the expectation value and this is consistent with the picture that, in quenched (i.e., non-supersymmetric) theory, the scalar potential is lifted by radiative corrections, suppressing quantum fluctuation of scalar fields. As discussed for the WT identity (3.16), if the supersymmetry is restored in the continuum limit, the expectation value $\langle\langle \mathcal{N}[a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}]\rangle\rangle$ is expected to become independent of ag as $a\to 0$. The behavior in figure 9 is more or less consistent with this expectation, although clearly we need further data at smaller values of ag to conclude this. In any case, interestingly, the expectation value appears to approach some finite number (in a unit of g^2) in the continuum limit after the renormalization (3.22). (Without the renormalization (the subtraction), there is a tendency that the expectation values grow as $a\to 0$.) The limiting value of $\langle\langle \mathcal{N}[a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}]\rangle\rangle$ at $a\to 0$ (while fixing Lg) in the figure itself has no direct physical meaning because it can freely be shifted by a further finite renormalization.

[16] We confirmed that the imaginary part is almost negligible (as it should be) and it is not plotted in the figure.

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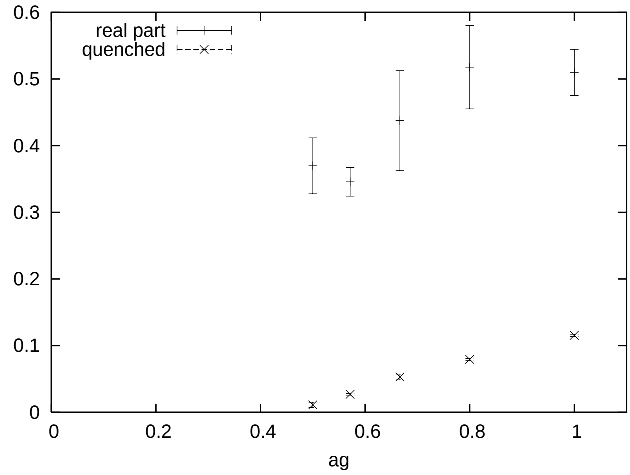

Figure 9: Expectation values of $\mathcal{N}[a^{-2}\,\mathrm{tr}\{\phi(x)\overline{\phi}(x)\}]$.

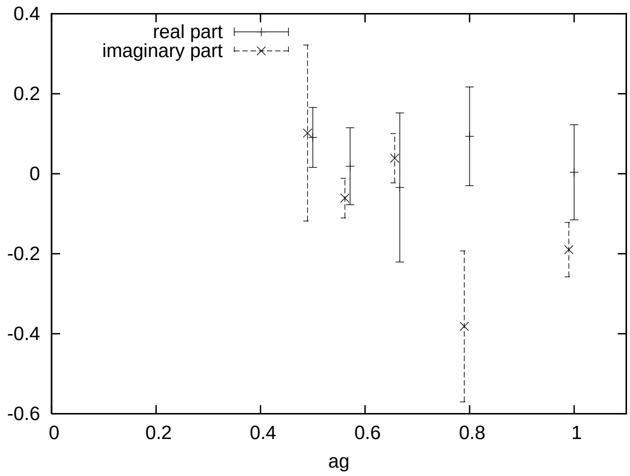

Figure 10: Expectation values of $a^{-2}\,\mathrm{tr}\{\phi(x)\phi(x)\}$.

However, the limiting value should depend on Lg and this dependence can be a non-trivial prediction. We need a much finer lattice, of course, for an extrapolation to the continuum.

In figure 10, we have plotted $\langle\!\langle a^{-2}\,\mathrm{tr}\{\phi(x)\phi(x)\}\rangle\!\rangle$. For this, a perturbative argument indicates that there is no need of renormalization. The result is clearly shows $\langle\!\langle a^{-2}\,\mathrm{tr}\{\phi(x)\phi(x)\}\rangle\!\rangle \sim 0$. This might be suggested from the fact that in two dimensions the global $\mathrm{U}(1)_R$ symmetry cannot be spontaneously broken, although this argument is not rigorous because we are studying a system in finite volume.

4. Conclusion

In this paper, we presented the results of our preliminary numerical study of Sugino's lattice formulation of the 2d $\mathcal{N} = (2, 2)$ SYM. By confirming WT identities associated with an exact fermionic symmetry of the formulation in fair accuracy, we infer that the re-weighting method for the dynamical fermions basically works in this two-dimensional system. On the other hand, although we could not conclude the restoration of full supersymmetry from the numerical results, all the results are consistent with the basic idea of a supersymmetry restoration. We computed also the expectation values of scalar bi-linear operators to illustrate the usefulness of this kind of lattice simulation.

In this paper, we did not try to measure any two-point correlation function or extended observables like Wilson loops, because it is clear that our lattice is too small to extract any useful information from such quantities. Interesting physics of this system is, of course, contained in these observables. For example, the most direct way to examine the restoration (and/or the spontaneous breaking) of supersymmetry is to study the mass spectra and two-point functions containing the supersymmetric current. For an interesting property of a two-point function that contains the $\mathrm{U}(1)_R$ current, see ref. [34]. Thanks to FermiQCD/MDP [33], our code is executable also on a large PC cluster without any change. Having obtained encouraging results in this paper, in the near future, we hope to report results of full-scale simulation using much larger lattice.

There exists a natural generalization of the present manifestly Q-invariant lattice formulation to the 2d $\mathcal{N} = (4, 4)$ SYM [1, 2] and to the 2d $\mathcal{N} = (8, 8)$ SYM (the second paper of ref. [15]). The latter theory is especially of interest as an effective theory that describes the dynamics of D1-brane. We do not find any real difficulty to set up the corresponding Monte Carlo simulation similar to that of the present paper. This is an interesting future problem.

Acknowledgments

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